

# RETURN OF $x + x^2y + z^2 + t^3 = 0$

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**ABSTRACT.** We develop techniques for computing the AK invariant of domains with arbitrary characteristic. As an example, we show that for any field  $\mathbf{k}$  the ring  $\mathbf{k}[X, Y, Z, T]/(X + X^2Y + Z^2 + T^3)$  is not isomorphic to a polynomial ring over  $\mathbf{k}$ .

## 1. INTRODUCTION

All rings in this paper are commutative with identity. Throughout this paper, let  $\mathbf{k}$  denote a field of arbitrary characteristic, and let  $\mathbf{k}^* = \mathbf{k} \setminus 0$ . For a ring  $A$ , let  $A^{[n]}$  denote the polynomial ring in  $n$  indeterminates over  $A$ . Let  $\mathbf{C}$  denote the complex number field.

The AK invariant was introduced by L. Makar-Limanov [M1] to show that P. Russell's [KR] threefold

$$x + x^2y + z^2 + t^3 = 0$$

over  $\mathbf{C}$  is not isomorphic to  $\mathbf{C}^3$ . The invariant was defined in that context as the intersection of the kernels of all locally nilpotent derivations on the coordinate ring of a variety. Since this first application, the AK invariant has been applied by several people (for instance [BM, Du, KKMR, M2, V]), mainly in the realm of affine algebraic geometry. To the author's knowledge, this work has all been conducted under the restriction of zero characteristic. Superficially, this is reasonable, since derivations don't behave as nicely on rings with prime characteristic  $p$ . The kernel of such a derivation doesn't convey the appropriate information in this setting, since it will always contain the  $p$ th power of every element.

Nevertheless, the apparent advantage to restricting the characteristic may be only a matter of perception. In the zero characteristic situation, locally nilpotent derivations on the coordinate ring correspond with algebraic additive group actions on the variety, and, unlike locally nilpotent derivations, these actions maintain their attractive properties in the prime characteristic setting. While some may find the zero characteristic setting more topologically natural or intuitive, others may see this view as a restriction which, like many restrictions, prevents the purest mathematical arguments from being made. Indeed, after R. Rentschler [R] provided a description of algebraic  $\mathbf{C}^+$ -actions on  $\mathbf{C}^2$ , M. Miyanishi [Mi1] demonstrated that Rentschler's theorem extends naturally without any technical difficulty to an affine plane of arbitrary characteristic, and many other papers appeared concurrently which studied such characteristic free techniques (e.g. the work of M. Miyanishi and Y. Nakai [Mi2, MN, N]). Also, after T. Fujita, M. Miyanishi and T. Sugie

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[F, MS] affirmatively solved the Zariski cancellation problem for the affine plane  $\mathbf{C}^2$ , P. Russell [Ru] gave a simplified treatment of their proof which erased the characteristic zero restriction.

The purpose of the present paper is to place the AK invariant in a characteristic free environment. We provide the definition and basic ideas, and we demonstrate computational techniques, closely following earlier treatments due to H. Derksen, O. Hadas, and L. Makar-Limanov [D, DHM, M1], which gain no simplification from the characteristic zero assumption. As an illustration, we compute the AK invariant for  $x + x^2y + z^2 + t^3 = 0$  over any field  $\mathbf{k}$  and obtain the same result which was originally found over  $\mathbf{C}$ . Similar efforts have been successfully applied toward a generalization of the Zariski cancellation problem [CM].

## 2. METHODS

**Exponential maps, the AK invariant, and locally finite iterative higher derivations.** Let  $A$  be a  $\mathbf{k}$ -algebra. Suppose  $\varphi : A \rightarrow A^{[1]}$  is a  $\mathbf{k}$ -algebra homomorphism. We write  $\varphi = \varphi_U : A \rightarrow A[U]$  if we wish to emphasize an indeterminate  $U$ . We say that  $\varphi$  is an *exponential map on  $A$*  if it satisfies the following two additional properties.

- (i)  $\varepsilon_0 \varphi_U$  is the identity on  $A$ , where  $\varepsilon_0 : A[U] \rightarrow A$  is evaluation at  $U = 0$ .
- (ii)  $\varphi_S \varphi_U = \varphi_{S+U}$ , where  $\varphi_S$  is extended by  $\varphi_S(U) = U$  to a homomorphism  $A[U] \rightarrow A[S, U]$ .

(When  $A$  is the coordinate ring of an affine variety  $\text{Spec}(A)$  over  $\mathbf{k}$ , the exponential maps on  $A$  correspond to algebraic actions of the additive group  $\mathbf{k}^+$  on  $\text{Spec}(A)$  [E, §9.5].) Define

$$A^\varphi = \{a \in A \mid \varphi(a) = a\},$$

a subalgebra of  $A$  called the *ring of  $\varphi$ -invariants*. Let  $\text{EXP}(A)$  denote the set of all exponential maps on  $A$ . We define the *AK invariant*, or *ring of absolute constants of  $A$*  as

$$\text{AK}(A) = \bigcap_{\varphi \in \text{EXP}(A)} A^\varphi.$$

This is a subalgebra of  $A$  which is preserved by isomorphism. Indeed, any isomorphism  $f : A \rightarrow B$  of  $\mathbf{k}$ -algebras restricts to an isomorphism  $f : \text{AK}(A) \rightarrow \text{AK}(B)$ . To understand this, observe that if  $\varphi \in \text{EXP}(A)$  then  $f\varphi f^{-1} \in \text{EXP}(B)$ . Remark that  $\text{AK}(A) = A$  if and only if the only exponential map on  $A$  is the standard inclusion  $\varphi(a) = a$  for all  $a \in A$ .

**Example 2.1.** By considering exponential maps of the form  $\varphi_i(X_j) = X_j + \delta_{ij}U$ , where  $\delta_{ij}$  is the Kronecker delta, one can see that  $\text{AK}(\mathbf{k}^{[n]}) = \mathbf{k}$  for each natural number  $n$ . When  $n = 1$ , this characterizes  $\mathbf{k}^{[1]}$  (see Lemma 2.4). However, if  $A$  is a domain with transcendence degree  $n \geq 2$  over  $\mathbf{k}$ , then  $\text{AK}(A) = \mathbf{k}$  does not imply that  $A \cong \mathbf{k}^{[n]}$  [BM].

It is often helpful to view a given  $\varphi \in \text{EXP}(A)$  as a sequence in the following way. For each  $a \in A$  and each natural number  $n$ , let  $D^n(a)$  denote the  $U^n$ -coefficient of  $\varphi(a)$ . Let  $D = \{D^0, D^1, D^2, \dots\}$ . To say that  $\varphi$  is a  $\mathbf{k}$ -algebra homomorphism is equivalent to saying that the sequence  $\{D^i(a)\}$  has finitely many nonzero elements

for each  $a \in A$ , that  $D^n : A \rightarrow A$  is  $\mathbf{k}$ -linear for each natural number  $n$ , and that the Leibniz rule

$$D^n(ab) = \sum_{i+j=n} D^i(a)D^j(b)$$

holds for all natural numbers  $n$  and all  $a, b \in A$ . The above properties (i) and (ii) of the exponential map  $\varphi$  translate into the following properties of  $D$ .

- (i')  $D^0$  is the identity map.
- (ii') (iterative property) For all natural numbers  $i, j$ ,

$$D^i D^j = \binom{i+j}{i} D^{i+j}.$$

Due to all of these properties, the collection  $D$  is called a *locally finite iterative higher derivation on  $A$* . More generally, a *higher derivation on  $A$*  is a collection  $D = \{D^i\}$  of  $\mathbf{k}$ -linear maps on  $A$  such that  $D^0$  is the identity and the above Leibniz rule holds. The notion of higher derivations is due to H. Hasse and F.K. Schmidt [HS]. When the characteristic of  $A$  is zero, each  $D^i$  is determined by  $D^1$ , which is a locally nilpotent derivation on  $A$ . In this case,  $\varphi = \exp(UD^1) = \sum_i \frac{1}{i!} (UD^1)^i$  and  $A^\varphi$  is the kernel of  $D^1$ .

The above discussion of exponential maps, locally finite iterative higher derivations, and the AK invariant makes sense more generally for any (not necessarily commutative) ring. However, we will not need this generality.

**Degree functions and related lemmas.** Given an exponential map  $\varphi : A \rightarrow A[U]$  on a domain  $A$  over  $\mathbf{k}$ , we can define the  $\varphi$ -degree of an element  $a \in A$  by  $\deg_\varphi(a) = \deg_U(\varphi(a))$  (where  $\deg_U(0) = -\infty$ ). Note that  $A^\varphi$  consists of all elements of  $A$  with non-positive  $\varphi$ -degree. The function  $\deg_\varphi$  is a degree function on  $A$ , i.e. it satisfies these two properties for all  $a, b \in A$ .

- (i)  $\deg_\varphi(ab) = \deg_\varphi(a) + \deg_\varphi(b)$ .
- (ii)  $\deg_\varphi(a + b) \leq \max\{\deg_\varphi(a), \deg_\varphi(b)\}$ .

Equipped with these notions, we now collect some useful facts.

**Lemma 2.2.** *Let  $\varphi$  be an exponential map on a domain  $A$  over  $\mathbf{k}$ . Let  $D = \{D^i\}$  be the locally finite iterative higher derivation associated to  $\varphi$ .*

- (a) *If  $a, b \in A$  such that  $ab \in A^\varphi \setminus 0$ , then  $a, b \in A^\varphi$ . In other words,  $A^\varphi$  is factorially closed in  $A$ .*
- (b)  *$A^\varphi$  is algebraically closed in  $A$ .*
- (c) *For each  $a \in A$ ,  $\deg_\varphi(D^i(a)) \leq \deg_\varphi(a) - i$ . In particular, if  $a \in A \setminus 0$  and  $n = \deg_\varphi(a)$ , then  $D^n(a) \in A^\varphi$ .*

*Proof.* (a): We have  $0 = \deg_\varphi(ab) = \deg_\varphi(a) + \deg_\varphi(b)$ , which implies that  $\deg_\varphi(a) = \deg_\varphi(b) = 0$ .

(b): If  $a \in A \setminus 0$  and  $c_n a^n + \cdots + c_1 a + c_0 = 0$  is a polynomial relation with minimal possible degree  $n \geq 1$ , where each  $c_i \in A^\varphi$  with  $c_0 \neq 0$ , then  $a(c_n a^{n-1} + \cdots + c_1) = -c_0 \in A^\varphi \setminus 0$ . By part (a),  $a \in A^\varphi$ .

(c): Use the iterative property of  $D$  to check that  $D^j(D^i(a)) = 0$  whenever  $j > \deg_\varphi(a) - i$ .  $\square$

**Lemma 2.3.** *Let  $\varphi$  be a nontrivial exponential map (i.e not the standard inclusion) on a domain  $A$  over  $\mathbf{k}$  with  $\text{char}(\mathbf{k}) = p \geq 0$ . Let  $x \in A$  with minimal positive  $\varphi$ -degree  $n$ .*

- (a)  $D^i(x) \in A^\varphi$  for each  $i \geq 1$ . Moreover,  $D^i(x) = 0$  whenever  $i \geq 1$  is not a power of  $p$ .
- (b) If  $a \in A \setminus 0$ , then  $n$  divides  $\deg_\varphi(a)$ .
- (c) Let  $c = D^n(x)$ . Then  $A$  is a subalgebra of  $A^\varphi[c^{-1}][x]$ , where  $A^\varphi[c^{-1}] \subseteq \text{Frac}(A^\varphi)$  is the localization of  $A^\varphi$  at  $c$ .
- (d) Let  $\text{trdeg}_{\mathbf{k}}$  denote transcendence degree over  $\mathbf{k}$ . If  $\text{trdeg}_{\mathbf{k}}(A)$  is finite, then  $\text{trdeg}_{\mathbf{k}}(A^\varphi) = \text{trdeg}_{\mathbf{k}}(A) - 1$ .

*Proof.* In proving parts (a) and (b) we will utilize the following fact. If  $p$  is prime and  $i = p^j q$  for some natural numbers  $i, j, q$ , then  $\binom{i}{p^j} \equiv q \pmod{p}$  [I, Lemma 5.1].

(a): By part (c) of Lemma 2.2,  $D^i(x) \in A^\varphi$  for all  $i \geq 1$ . If  $p = 0$  then  $n = 1$ , for given any element in  $A \setminus A^\varphi$  we can find an element with  $\varphi$ -degree 1 by applying the locally nilpotent derivation  $D^1$  sufficiently many times. In this case, the second statement is immediate. Suppose now that  $p$  is prime and that  $i > 1$  is not a power of  $p$ , say  $i = p^j q$ , where  $j$  is a nonnegative integer and  $q \geq 2$  is an integer not divisible by  $p$ . Then  $D^{i-p^j}(x) \in A^\varphi$  and

$$0 = D^{p^j} D^{i-p^j}(x) = \binom{i}{p^j} D^i(x) = q D^i(x).$$

We can divide by  $q$  to conclude that  $D^i(x) = 0$ .

(b): Again if  $p = 0$  then  $n = 1$  and the claim is obvious. Assume that  $p$  is prime. By part (a) we have  $n = p^m$  for some integer  $m \geq 0$ . If  $m = 0$ , the claim is immediate. Assume that  $m > 0$ . Let  $d = \deg_\varphi(a)$ . Suppose that  $p$  does not divide  $d$ . By part (c) of Lemma 2.2,  $\deg_\varphi(D^{d-1}(a)) \leq 1$ . Now,  $D^1 D^{d-1}(a) = d D^d(a) \neq 0$ . So  $\deg_\varphi(D^{d-1}(a)) = 1 < n$ , contradicting the minimality of  $n$ . Hence we can write  $d = p^k d_1$  with  $k \geq 1$  and  $d_1$  not divisible by  $p$ . Making a similar computation,  $D^{p^k} D^{d-p^k}(a) = d_1 D^d(a) \neq 0$ . This implies that  $\deg_\varphi(D^{d-p^k}(a)) = p^k$ . Since  $n = p^m$  is minimal, we must have  $k \geq m$ , and so  $n$  divides  $d$ .

(c): Let  $a \in A \setminus 0$ . By part (b) we can write  $\deg_\varphi(a) = ln$  for some natural number  $l$ . If  $l = 0$  then  $a \in A^\varphi$  and we are done. We use induction on  $l > 0$ . Elements  $c^l a$  and  $D^{ln}(a)x^l$  both have  $\varphi$ -degree  $ln$ . Let us check that  $D^{ln}(c^l a) = D^{ln}(D^{ln}(a)x^l)$ . First,  $D^{ln}(c^l a) = c^l D^{ln}(a)$  by the Leibniz rule and because  $c^l$  is  $\varphi$ -invariant. Secondly, since  $D^{ln}(x^l) = D^n(x)^l = c^l$  and  $D^{ln}(a)$  is  $\varphi$ -invariant, we see that  $D^{ln}(D^{ln}(a)x^l) = c^l D^{ln}(a)$  as well. (Remark: Though the equality  $D^{ln}(x^l) = D^n(x)^l$  does follow from the Leibniz rule, it may be more immediately observed as follows.  $D^n(x)$  is the leading  $U$ -coefficient of  $\varphi(x)$ , and  $\varphi$  is a homomorphism. Hence the leading  $U$ -coefficient of  $\varphi(x^l)$  is also that of  $\varphi(x)^l$ .) Therefore, the element  $y = c^l a - D^{ln}(a)x^l$  has  $\varphi$ -degree less than  $ln$  and hence less than or equal to  $(l-1)n$ . By the inductive hypothesis,  $y \in A^\varphi[c^{-1}][x]$ . So  $a = c^{-l}(y + D^{ln}(a)x^l) \in A^\varphi[c^{-1}][x]$ .

(d): This is immediate from part (c), together with part (b) of Lemma 2.2 which states that  $A^\varphi$  is algebraically closed in  $A$ .  $\square$

**Lemma 2.4.** *Let  $A$  be a domain over  $\mathbf{k}$  with  $\text{trdeg}_{\mathbf{k}}(A) = 1$ . Then  $\text{AK}(A) = \mathbf{k}$  if and only if  $A \cong \mathbf{k}^{[1]}$ . Otherwise,  $\text{AK}(A) = A$ .*

*Proof.* To see that  $\text{AK}(\mathbf{k}[X]) = \mathbf{k}$  (where  $X$  is an indeterminate), observe that  $\psi(X) = X + U$  defines an exponential map on  $\mathbf{k}[X]$  with ring of  $\psi$ -invariants  $\mathbf{k}$ . Suppose  $\text{AK}(A) \neq A$ . Let  $\varphi \in \text{EXP}(A)$  be nontrivial. Part (b) of Lemma 2.2 implies that  $A^\varphi = \mathbf{k}$ . By part (c) of Lemma 2.3,  $A \subseteq \mathbf{k}[x]$  for some  $x \in A$  with minimal positive  $\varphi$ -degree. So  $A = \mathbf{k}[x]$ .  $\square$

Thus  $\mathbf{k}^{[1]}$  is the only transcendence degree 1 domain over  $\mathbf{k}$  which admits non-trivial exponential maps.

**Lemma 2.5.** *Let  $\varphi$  be an exponential map on a domain  $A$  over  $\mathbf{k}$ . Extend  $\varphi$  to a homomorphism  $\varphi : \text{Frac}(A) \rightarrow \text{Frac}(A)(U)$  by  $\varphi(ab^{-1}) = \varphi(a)\varphi(b)^{-1}$ , and let  $\text{Frac}(A)^\varphi = \{f \in \text{Frac}(A) \mid \varphi(f) = f\}$ . Then  $\text{Frac}(A)^\varphi = \text{Frac}(A^\varphi)$ .*

*Proof.* It is clear that  $\text{Frac}(A^\varphi) \subseteq \text{Frac}(A)^\varphi$ . Suppose that  $a, b \in A \setminus 0$ , such that  $ab^{-1} \in \text{Frac}(A)^\varphi$ . Then

$$ab^{-1} = \varphi(ab^{-1}) = \varphi(a)\varphi(b)^{-1} = \left( \sum_i D^i(a)U^i \right) \left( \sum_i D^i(b)U^i \right)^{-1},$$

and so

$$\sum_i aD^i(b)U^i = \sum_i bD^i(a)U^i.$$

Compare the leading coefficients to see that  $a$  and  $b$  have the same  $\varphi$ -degree, say  $n$ , and  $ab^{-1} = D^n(a)D^n(b)^{-1} \in \text{Frac}(A^\varphi)$ .  $\square$

Let  $A$  be a domain over  $\mathbf{k}$ , and let  $\varphi \in \text{EXP}(A)$ . Let  $B$  be the domain over  $\text{Frac}(A^\varphi)$  obtained by localizing  $A$  at the multiplicative set  $A^\varphi \setminus 0$ . By Lemma 2.5,  $\varphi$  extends to a  $\text{Frac}(A^\varphi)$ -homomorphism  $\varphi : B \rightarrow B[U]$  which is an exponential map on  $B$  with ring of invariants  $\text{Frac}(A^\varphi)$ .

**Homogenization of an exponential map.** Let  $A$  be a domain over  $\mathbf{k}$ . Let  $\mathbf{Z}$  denote the integers. Suppose that  $A$  has a  $\mathbf{Z}$ -filtration  $\{A_n\}$ . This means that  $A$  is the union of linear subspaces  $A_n$  with these properties:  $A_i \subseteq A_j$  whenever  $i \leq j$ , and  $A_i \cdot A_j \subseteq A_{i+j}$  for all  $i, j \in \mathbf{Z}$ . Additionally, suppose that

$$(A_i \setminus A_{i-1}) \cdot (A_j \setminus A_{j-1}) \subseteq A_{i+j} \setminus A_{i+j-1}$$

for all  $i, j \in \mathbf{Z}$ . This will be the case if the filtration is induced by a degree function. Suppose also that  $\chi$  is a set of generators for  $A$  over  $\mathbf{k}$  with the following property: if  $a \in A_i \setminus A_{i-1}$  then we can write  $a = \sum_I c_I \mathbf{x}^I$ , a summation of monomials  $c_I \mathbf{x}^I$  built from  $\chi$  which are all contained in  $A_i$ . This is not an unreasonable property. It merely asserts some homogeneity on the generating set  $\chi$ .

Given  $a \in A \setminus 0$  there exists  $i \in \mathbf{Z}$  for which  $a \in A_i \setminus A_{i-1}$ . Write

$$\bar{a} = a + A_{i-1} \in A_i/A_{i-1},$$

the *top part* of  $a$ . We can construct a graded  $\mathbf{k}$ -algebra

$$\text{gr}(A) = \bigoplus_{n \in \mathbf{Z}} A_n/A_{n-1}.$$

Addition on  $\text{gr}(A)$  is given by its vector space structure. Given  $\bar{a} = a + A_{i-1}$  and  $\bar{b} = b + A_{j-1}$ , define  $\bar{a}\bar{b} = ab + A_{i+j-1}$ . Note that  $\overline{\bar{a}\bar{b}} = \overline{\bar{a}}\overline{\bar{b}}$ . Extend this multiplication to all of  $\text{gr}(A)$  by the distributive law. By our assumption on the filtration,  $\text{gr}(A)$  is a domain. Also,  $\text{gr}(A)$  is generated by the top parts of the elements of  $\chi$ . Therefore, if  $\chi$  is a finite set then  $\text{gr}(A)$  is an affine domain.

Let  $\text{grdeg}$  be the degree function induced by the grading on  $\text{gr}(A)$ . By assigning a value to  $\text{grdeg}(U)$  for an indeterminate  $U$ , we can extend the grading on  $\text{gr}(A)$  to  $\text{gr}(A)[U]$ . Given an exponential map  $\varphi : A \rightarrow A[U]$  on  $A$ , the goal is to obtain an exponential map  $\bar{\varphi}$  on  $\text{gr}(A)$ . For  $a \in A$ , let  $\text{grdeg}(a)$  denote  $\text{grdeg}(\bar{a})$ . Note that

$\text{grdeg}(\bar{a}) = i$  if and only if  $a \in A_i \setminus A_{i-1}$ . Consequently,  $\text{grdeg}$  can also be viewed as a degree function on  $A$  and on  $A[U]$  once the value of  $\text{grdeg}(U)$  is determined.

Define

$$(\star) \quad \text{grdeg}(U) = \min \left\{ \frac{\text{grdeg}(x) - \text{grdeg}(D^i(x))}{i} \mid x \in \chi, i \in \mathbf{Z}^+ \right\}.$$

Let us assume now that  $\text{grdeg}(U)$  does exist, i.e. is a rational number. This will indeed occur whenever  $\chi$  is a finite set, as will be the case with our example of interest. If  $x \in \chi$  and  $n$  is a natural number, then  $\text{grdeg}(D^n(x)U^n) \leq \text{grdeg}(x)$  by our choice of  $\text{grdeg}(U)$ . From this it follows by straightforward calculation that  $\text{grdeg}(D^n(a)U^n) \leq \text{grdeg}(a)$  for all  $a \in A$  and all natural numbers  $n$ . (Here we use the homogeneity assumption imposed on  $\chi$ .) The reader can easily work out the details or refer to [C]. Note that this inequality is sharp since

$$\text{grdeg}(U) = \frac{1}{n}(\text{grdeg}(x) - \text{grdeg}(D^n(x)))$$

for some  $x \in \chi$  and some positive integer  $n$  (and also since  $D^0(a) = a$  for all  $a \in A$ ).

For  $a \in A$ , let

$$S(a) = \{n \mid \text{grdeg}(D^n(a)) + n \text{grdeg}(U) = \text{grdeg}(a)\}.$$

Define

$$\bar{\varphi}(\bar{a}) = \sum_{n \in S(a)} \overline{D^n(a)} U^n$$

and extend this linearly to define  $\bar{\varphi} : \text{gr}(A) \rightarrow \text{gr}(A)[U]$ , the *homogenization* or *top part* of  $\varphi$ . One can verify that  $\bar{\varphi}$  is an exponential map on  $\text{gr}(A)$ . Refer to [DHM] for the case  $A = \mathbf{k}^{[n]}$ . The proof of the general case is symbolically identical. Let  $\overline{A^\varphi}$  denote the domain generated by the top parts of all elements in  $A^\varphi$ . The end result is

**Theorem 2.6** (H. Derksen, O. Hadas, L. Makar-Limanov [DHM]). *Let  $A$  be a domain over  $\mathbf{k}$  with  $\mathbf{Z}$ -filtration  $\{A_n\}$  such that  $(A_i \setminus A_{i-1}) \cdot (A_j \setminus A_{j-1}) \subseteq A_{i+j} \setminus A_{i+j-1}$  for all  $i, j \in \mathbf{Z}$ . Let  $\varphi$  be a nontrivial exponential map on  $A$ . Assume that  $\text{grdeg}(U)$  exists as defined above. Then  $\bar{\varphi}$  as defined above is a nontrivial exponential map on  $\text{gr}(A)$ . Moreover,  $\overline{A^\varphi}$  is contained in  $\text{gr}(A)^{\bar{\varphi}}$ .*

An important special case of homogenization is when  $A$  itself is graded. Then we can filter  $A$  so that  $\text{gr}(A)$  is canonically isomorphic to  $A$ , and we can choose  $\chi$  to be a set of homogeneous generators of  $A$ . In this case the top part of  $\varphi$  is a nontrivial exponential map on  $A$  (assuming  $\text{grdeg}(U)$  exists).

**Example 2.7.** Let  $A = \mathbf{k}[X, Y]$ , where  $\text{char}(\mathbf{k}) = p$ , prime. Define  $\varphi \in \text{EXP}(A)$  by  $\varphi(X) = X$  and  $\varphi(Y) = Y + U + XU^p$ . We can grade  $A$  by setting  $\text{grdeg}(X) = \alpha$  and  $\text{grdeg}(Y) = \beta$  (with  $\text{grdeg}(\lambda) = 0$  for all  $\lambda \in \mathbf{k}^*$ , and  $\text{grdeg}(0) = -\infty$ ). Since  $\text{grdeg}(D^i(X)) = -\infty$  for all  $i \geq 1$ ,  $X$  will not contribute to the value of  $\text{grdeg}(U)$ . Therefore,

$$\begin{aligned} \text{grdeg}(U) &= \min \left\{ \frac{\text{grdeg}(Y) - \text{grdeg}(1)}{1}, \frac{\text{grdeg}(Y) - \text{grdeg}(X)}{p} \right\} \\ &= \min \left\{ \beta, \frac{\beta - \alpha}{p} \right\}. \end{aligned}$$

In any case,  $\overline{\varphi}(X) = X$ . If  $\beta < \frac{1}{p}(\beta - \alpha)$  then  $\text{grdeg}(U) = \beta$  and  $\overline{\varphi}(Y) = Y + U$ . If  $\beta = \frac{1}{p}(\beta - \alpha)$  then  $\text{grdeg}(U) = \beta$  and  $\overline{\varphi}(Y) = \varphi(Y)$ . If  $\beta > \frac{1}{p}(\beta - \alpha)$  then  $\text{grdeg}(U) = \frac{1}{p}(\beta - \alpha)$  and  $\overline{\varphi}(Y) = Y + XU^p$ .

### 3. THE RUSSELL HYPERSURFACE

Let  $R = \mathbf{k}[X, Y, Z, T]/(X + X^2Y + Z^2 + T^3)$ . If we wish to emphasize a choice of  $\mathbf{k}$ , we write  $R = R_{\mathbf{k}}$ . Let  $x, y, z, t \in R$  denote the cosets of  $X, Y, Z, T$ , respectively. We shall prove

**Theorem 3.1.** *Suppose  $\mathbf{k}$  is algebraically closed. Then  $\text{AK}(R) = \mathbf{k}[x]$ .*

If  $\mathbf{k}$  is any field, then  $\text{AK}(\mathbf{k}^{[3]}) = \mathbf{k}$ . Also, if  $\overline{\mathbf{k}}$  is an algebraic closure of  $\mathbf{k}$ , then  $R_{\mathbf{k}} \otimes_{\mathbf{k}} \overline{\mathbf{k}} = R_{\overline{\mathbf{k}}}$ . These observations lead immediately to

**Corollary 3.2.**  *$R \not\cong \mathbf{k}^{[3]}$  for any  $\mathbf{k}$ .*

To prove Theorem 3.1, let us start with

**Lemma 3.3.** *Suppose  $\mathbf{k}$  is algebraically closed. Then  $x \in \text{AK}(R)$ .*

*Proof.* Suppose that  $\varphi : R \rightarrow R[U]$  is a nontrivial exponential map on  $R$ . We want to show that  $x \in R^{\varphi}$ . Let us consider  $R$  as a subalgebra of  $\mathbf{k}[x, x^{-1}, z, t]$  with

$$y = -x^{-2}(x + z^2 + t^3).$$

Introduce a degree function  $w_1$  on  $\mathbf{k}[x, x^{-1}, z, t]$  by

$$\begin{aligned} w_1(x) &= -1, \\ w_1(z) &= 0, \\ w_1(t) &= 0 \end{aligned}$$

(with  $w_1(\lambda) = 0$  for all  $\lambda \in \mathbf{k}^*$  and  $w_1(0) = -\infty$ ). Then

$$w_1(y) = 2.$$

This induces a  $\mathbf{Z}$ -filtration  $\{R_i\}$  on  $R$ , where  $R_i$  consists of all  $r \in R$  with  $w_1(r) \leq i$ . Passing to top parts,  $\overline{y} = -\overline{x}^{-2}(\overline{z}^2 + \overline{t}^3)$ . So the corresponding graded domain  $\text{gr}(R)$  is generated by  $\overline{x}, \overline{y}, \overline{z}, \overline{t}$  and subject to the relation  $\overline{x}^2\overline{y} + \overline{z}^2 + \overline{t}^3 = 0$ . Let us write  $x, y, z, t$  in place of  $\overline{x}, \overline{y}, \overline{z}, \overline{t}$ , respectively. Then

$$\text{gr}(R) = \mathbf{k}[x, y, z, t]/(x^2y + z^2 + t^3).$$

For a first step we show

**Sublemma 3.4.**  *$R^{\varphi} \subseteq \mathbf{k}[x, z, t]$ .*

*Proof.* Suppose that  $f \in R^{\varphi}$  and  $f \notin \mathbf{k}[x, z, t]$ . It is clear that the value  $\text{grdeg}(U)$  as defined by formula  $(\star)$  exists for our filtration, since  $R$  is finitely generated by  $\chi = \{x, y, z, t\}$ . By Theorem 2.6,  $\varphi$  induces a nontrivial exponential map  $\overline{\varphi}$  on  $\text{gr}(R)$  with  $\overline{f} \in \text{gr}(R)^{\overline{\varphi}}$ . We can write

$$\overline{f} = x^a y^b g(z, t)$$

for some natural numbers  $a$  and  $b$  and some polynomial  $g(z, t) \in \mathbf{k}[z, t]$ . By our assumption on  $f$ , we know  $b$  must be positive. We can assume that  $a = 0$  or  $a = 1$ , since a factor  $x^2y$  of  $\overline{f}$  can be absorbed into  $g(z, t)$  by the relation on  $\text{gr}(R)$ . If  $a = 1$ , then we can replace  $f$  by  $f^2$ , and so we may assume that  $a = 0$ . So now

$\overline{f} = y^b g(z, t)$ . Also, since  $\text{trdeg}_{\mathbf{k}}(R^\varphi) = 2$  (part (d) of Lemma 2.3), we can assume that  $g(z, t)$  is not a constant polynomial by replacing  $f$  if necessary.

Since  $\text{gr}(R)^\varphi$  is factorially closed (part (a) of Lemma 2.2), both  $y$  and  $g(z, t)$  belong to  $\text{gr}(R)^\varphi$ . Let us introduce a new grading on  $\text{gr}(R)$  by  $w_2(x) = 6$ ,  $w_2(y) = -6$ ,  $w_2(z) = 3$ , and  $w_2(t) = 2$ . The corresponding graded domain (still call it  $\text{gr}(R)$ ) is again isomorphic to  $\mathbf{k}[x, y, z, t]/(x^2y + z^2 + t^3)$ , and let us continue to write  $x, y, z, t$  in place of  $\overline{x}, \overline{y}, \overline{z}, \overline{t}$ . Under this new grading, we can write

$$g(z, t) = \lambda z^n t^m \prod_i (z^2 + \mu_i t^3)$$

for some natural numbers  $n$  and  $m$  and for some  $\lambda, \mu_i \in \mathbf{k}^*$ .

The next step is to show that neither  $z$  nor  $t$  can be  $\varphi$ -invariant. Suppose that  $z \in \text{gr}(R)^\varphi$ . Then  $\text{gr}(R)^\varphi = \mathbf{k}[y, z]$ . By virtue of the remark following Lemma 2.5,  $\varphi$  induces a nontrivial exponential map on the domain

$$\mathbf{k}(y, z)[x, t]/(x^2y + z^2 + t^3).$$

But this domain has transcendence degree 1 over  $\mathbf{k}(y, z)$  and is not isomorphic to  $\mathbf{k}(y, z)^{[1]}$ . This contradicts Lemma 2.4. Thus  $z \notin \text{gr}(R)^\varphi$ . Suppose now that  $t \in \text{gr}(R)^\varphi$ . Then  $\text{gr}(R)^\varphi = \mathbf{k}[y, t]$  and just as with  $z$  we can obtain a nontrivial exponential map on the domain

$$\mathbf{k}(y, t)[x, z]/(x^2y + z^2 + t^3).$$

Again, this domain has transcendence degree 1 over  $\mathbf{k}(y, t)$  but is not isomorphic to  $\mathbf{k}(y, t)^{[1]}$ , contradicting Lemma 2.4. So  $t \notin \text{gr}(R)^\varphi$  as well. Since  $\text{gr}(R)^\varphi$  is factorially closed,  $n = 0$  and  $m = 0$  in the above factorization of  $g(z, t)$ .

It remains to consider the factors of  $g(z, t)$  of the form  $z^2 + \mu_i t^3$ . Ignoring multiplicity, there can only be one such factor. For given two factors of this type, their difference belongs to  $\text{gr}(R)^\varphi$  from which we conclude that both  $z^2$  and  $t^3$  belong to  $\text{gr}(R)^\varphi$ , which we have just shown to be impossible. Also,  $z^2 + t^3$  cannot be a factor of  $g(z, t)$ , since otherwise we would have  $x^2y = -(z^2 + t^3) \in \text{gr}(R)^\varphi$ , which in turn implies that  $\text{gr}(R)^\varphi = \text{gr}(R)$ , contradicting the nontriviality of  $\varphi$ . We can therefore write  $g(z, t)$  as

$$g(z, t) = \lambda(z^2 + \mu t^3)^k$$

for some positive integer  $k$  and some  $\lambda, \mu \in \mathbf{k}^*$  with  $\mu \neq 1$ . We will use the same trick that worked for  $z$  and  $t$ . Let  $S$  be the domain which results from localizing  $\text{gr}(R)$  at  $\text{gr}(R)^\varphi \setminus 0$ . Recall that  $y$  and  $z^2 + \mu t^3$  belong to  $\text{gr}(R)^\varphi$ . Note that we can rewrite  $x^2y + z^2 + t^3 = 0$  as the relation

$$x^2y + (1 - \mu)t^3 + (z^2 + \mu t^3) = 0$$

over  $\text{Frac}(\text{gr}(R)^\varphi)$ . From this we can see that  $S$  has transcendence degree 1 over  $\text{Frac}(\text{gr}(R)^\varphi)$  but is not isomorphic to  $\text{Frac}(\text{gr}(R)^\varphi)^{[1]}$ . We can extend  $\varphi$  to a nontrivial exponential map on  $S$  over  $\text{Frac}(\text{gr}(R)^\varphi)$  in the manner described after Lemma 2.5. This once again contradicts Lemma 2.4. We have now exhausted all possibilities. To avoid a contradiction, we must have  $f \in \mathbf{k}[x, z, t]$ . So  $R^\varphi \subseteq \mathbf{k}[x, z, t]$ .  $\square$

Continuing with the proof of Lemma 3.3, we are now in position to show that  $x \in R^\varphi$ . Suppose that it is not the case. If  $f \in R^\varphi \subseteq \mathbf{k}[x, z, t]$ , write  $f = x f_1(x, z, t) + f_2(z, t)$ . Then  $f_2 \neq 0$  since  $x \notin R^\varphi$ . Again consider  $\text{gr}(R)$  given by



$w_1$ . Now  $w_1(xf_1(x, z, t))$  is negative, while  $w_1(f_2(z, t)) = 0$ , and so  $\overline{f} = f_2(z, t) \in \text{gr}(R)^{\overline{\varphi}}$ . Let  $g \in R^{\varphi}$  be algebraically independent with  $f$  over  $\mathbf{k}$ . (Recall that  $\text{trdeg}_{\mathbf{k}}(R^{\varphi}) = 2$  by part (d) of Lemma 2.3.) We write  $g = xg_1(x, z, t) + g_2(z, t)$ , where  $0 \neq g_2(z, t) = \overline{g}$ . Suppose that  $f_2$  and  $g_2$  are algebraically dependent over  $\mathbf{k}$ , say  $P(f_2, g_2) = 0$ . Then  $P(f, g)$  is a nonzero element of  $R^{\varphi}$ , but  $P(f, g)$  is divisible by  $x$ . This implies that  $x \in R^{\varphi}$ , contrary to our assumption. Hence it must be that  $f_2$  and  $g_2$  are algebraically independent over  $\mathbf{k}$ . Thus  $\text{gr}(R)^{\overline{\varphi}}$  contains two algebraically independent elements of  $\mathbf{k}[z, t]$ . Since  $\text{gr}(R)^{\overline{\varphi}}$  is algebraically closed in  $\text{gr}(R)$  (part (b) of Lemma 2.2), we deduce that  $\text{gr}(R)^{\overline{\varphi}} = \mathbf{k}[z, t]$ . Now  $x^2y = -z^2 - t^3 \in \text{gr}(R)^{\overline{\varphi}}$ , and this implies that  $x, y \in \text{gr}(R)^{\overline{\varphi}}$ . But then  $\overline{\varphi}$  is trivial. This contradicts our assumption that  $x \notin R^{\varphi}$ . So  $x \in R^{\varphi}$  for every  $\varphi \in \text{EXP}(R)$ , and the lemma is finally proved.  $\square$

*Proof of Theorem 3.1.* We know that  $\mathbf{k}[x] \subseteq \text{AK}(R)$ . To show the reverse containment, we consider two maps  $\varphi_1$  and  $\varphi_2$  which one easily verifies to be exponential maps on  $R$ . Define  $\varphi_1 : R \rightarrow R[U]$  by

$$\begin{aligned}\varphi_1(x) &= x, \\ \varphi_1(y) &= y + 2zU - x^2U^2, \\ \varphi_1(z) &= z - x^2U, \\ \varphi_1(t) &= t.\end{aligned}$$

The ring of  $\varphi_1$ -invariants is  $\mathbf{k}[x, t]$ . Define  $\varphi_2 : R \rightarrow R[U]$  by

$$\begin{aligned}\varphi_2(x) &= x, \\ \varphi_2(y) &= y + 3t^2U - 3x^2tU^2 + x^4U^3, \\ \varphi_2(z) &= z, \\ \varphi_2(t) &= t - x^2U.\end{aligned}$$

The ring of  $\varphi_2$ -invariants is  $\mathbf{k}[x, z]$ . So  $\text{AK}(R)$  is contained in the intersection of these two rings, that being  $\mathbf{k}[x]$ .  $\square$

As a final remark, L. Makar-Limanov [M3] has recently taken a simplified approach (again very similar to the proof given here) to showing that  $\text{AK}(R_{\mathbf{C}}) = \mathbf{C}[x]$ . This proof uses the following fact.

**Lemma 3.5** (see [M3]). *Let  $A$  be a domain with characteristic zero. Let  $n$  and  $m$  be natural numbers both at least 2. Let  $\varphi \in \text{EXP}(A)$  and  $c_1, c_2 \in A^{\varphi} \setminus 0$ . If  $a, b \in A$  such that  $c_1a^n + c_2b^m \in A^{\varphi} \setminus 0$ , then  $a, b \in A^{\varphi}$ .*

In fact, this lemma is still true when  $A$  has prime characteristic  $p$ , under the additional necessary hypothesis that neither  $n$  nor  $m$  are powers of  $p$  [C]. This lemma can then replace the many times that we used Lemma 2.5 to contradict Lemma 2.4. But because of the extra assumption needed on Lemma 3.5, this method fails when the characteristic of  $R$  is 2 or 3.

#### REFERENCES

- [BM] T. Bandman and L. Makar-Limanov, Affine surfaces with  $\text{AK}(S) = \mathbf{C}$ , *Michigan Math. J.* **49**(2001), 567-582.
- [C] A. Crachiola, *On the AK invariant of certain domains*, Ph.D. thesis, Wayne State University, 2004.

- [CM] A. Crachiola and L. Makar-Limanov, On the rigidity of small domains, preprint, 2004.
- [D] H. Derksen, More on the hypersurface  $x + x^2y + z^2 + t^3 = 0$  in  $\mathbf{C}^4$ , preprint, 1995.
- [DHM] H. Derksen, O. Hadas, and L. Makar-Limanov, Newton polytopes of invariants of additive group actions, *J. Pure Appl. Algebra* **156**(2001), 187-197.
- [Du] A. Dubouloz, Generalized Danielewski surfaces, preprint, 2004.
- [E] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progr. Math., Vol. 190, Birkhäuser Verlag, Basel, 2000.
- [F] T. Fujita, On Zariski problem, *Proc. Japan Acad. Ser. A Math. Sci.* **55**(1979), 106-110.
- [HS] H. Hasse and F.K. Schmidt, Noch eine Begründung der Theorie der höheren Differentialquotienten in einem algebraischen Functionenkörper einer Unbestimmten, *J. Reine Angew. Math.* **177**(1937), 215-237.
- [I] I.M. Isaacs, *Algebra, a graduate course*, Brooks/Cole, Pacific Grove, California, 1994.
- [KKMR] S. Kaliman, M. Koras, L. Makar-Limanov, and P. Russell,  $\mathbf{C}^*$ -actions on  $\mathbf{C}^3$  are linearizable, *Electron. Res. Announc. Amer. Math. Soc.* **3**(1997), 63-71.
- [KR] M. Koras and P. Russell, Contractible threefolds and  $\mathbf{C}^*$ -actions on  $\mathbf{C}^3$ , *J. Algebraic Geometry* **6**(1997), no. 4, 671-695.
- [M1] L. Makar-Limanov, On the hypersurface  $x + x^2y + z^2 + t^3 = 0$  in  $\mathbf{C}^4$ , or a  $\mathbf{C}^3$ -like threefold which is not  $\mathbf{C}^3$ , *Israel J. Math.* **96**(1996), 419-429.
- [M2] L. Makar-Limanov, On the group of automorphisms of a surface  $x^ny = P(z)$ , *Israel J. Math.* **121**(2001), 113-123.
- [M3] L. Makar-Limanov, Again  $x + x^2y + z^2 + t^3 = 0$ , preprint, 2004.
- [Mi1] M. Miyanishi,  $G_a$ -action of the affine plane, *Nagoya Math. J.* **41**(1971), 97-100.
- [Mi2] M. Miyanishi, Some remarks on polynomial rings, *Osaka J. Math.* **10**(1973), 617-624.
- [MN] M. Miyanishi and Y. Nakai, Some remarks on strongly invariant rings, *Osaka J. Math.* **12**(1975), 1-17.
- [MS] M. Miyanishi and T. Sugie, Affine surfaces containing cylinderlike open sets, *J. Math. Kyoto Univ.* **20**(1980), 11-42.
- [N] Y. Nakai, On locally finite iterative higher derivations, *Osaka J. Math.* **15**(1978), 655-662.
- [R] R. Rentschler, Opérations du groupe additif sur le plane affine, *C.R. Acad. Sci. Paris* **267**(1968), 384-387.
- [Ru] P. Russell, On affine-ruled rational surfaces, *Math. Ann.* **255**(1981), 287-302.
- [V] M. Villa, Some new examples of 4-dimensional algebraic exotic structures, preprint, 2003.

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